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An asymptotic approach to second-kind similarity solutions of the modified porous-medium equation

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Abstract. The problem of a spreading ground-water mound of liquid in a porous medium, situated on an impermeable horizontal solid layer is revisited. The mathematical formulation for this problem is given by the modified porous medium equation. A global condition in form of an energy integral is derived, describing the loss of liquid in the porous medium. This yields the necessary condition that enables the aymptotic derivation of the similarity exponents for the similarity solution of second kind. The method developed here, is further applied to the corresponding dipole problem, when instead of an energy integral another conservation law, the first moment integral, is considered.

Key words: asymptotics, Lie group methods, porous-medium equation, similarity solutions

I dedicate this article to the memory of Julian D. Cole,
who has been a great inspiration to me
and to the Applied Mathematics community in general for his
profound impact on the development of Perturbation Methods.

1. Introduction

We revisit the problem, previously discussed in [1,2], [3, pp. 52–54] and references therein, of a porous medium that is partially filled with a radially symmetric mound of liquid bounded by $u(r,t)$. Under the action of gravity the liquid spreads and displaces the gas outside the mound at some points, while at other points, pores that were previously filled with liquid are being occupied by the gas. In the mound at some (initial) time t_0 the liquid saturation is assumed to be equal to σ_+. Due to capillary forces some liquid remains in the pores having a residual saturation σ_-, see Figure 1. For slow fluid motion in the porous medium the vertical pressure in the mound can be assumed to obey the hydrostatic law $p=\rho g\,(u-z)$, where ρ is the liquid density and g the gravitational constant. Then, according to Darcy's law the flux of liquid through a cylindrical surface of area $2\pi r u$ is

$$v=-\frac{\kappa}{\mu}\,2\pi\, r\, u\,\frac{\partial p}{\partial r}=-\frac{\kappa\rho g\pi}{\mu}\,r\,\frac{\partial u^2}{\partial r}\,, \tag{1.1}$$

where κ is the permeability of the porous medium and μ the viscosity of the liquid, see *e.g.* the classic book [4] for details. Since the change in the flux of the liquid is equal to the rate of change in the volume of the liquid mound, due to the decreasing saturation from σ_+ to σ_-, one obtains

$$2\pi r m(\sigma_+-\sigma_-)\frac{\partial u}{\partial t}=\frac{\kappa\rho g\pi}{\mu}\frac{\partial}{\partial r}\left(r\frac{\partial u^2}{\partial r}\right), \tag{1.2}$$

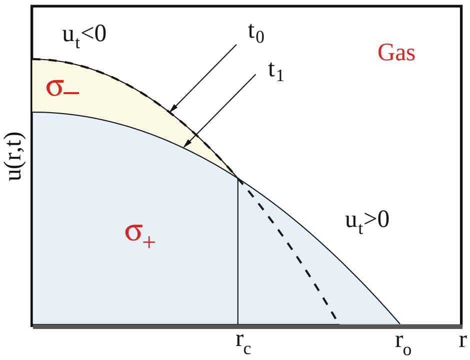

Figure 1. The spreading ground-water mound in porous medium.

where m is the volume of the porous medium, occupied by the pores. This occurs for $r \in [0, r_c)$, *i.e.*, where $\partial u/\partial t < 0$. Beyond r_c the liquid saturation is increasing from zero to σ_+, *i.e.*, where $\partial u/\partial t > 0$ the governing equation is

$$2\pi r m\sigma_+ \frac{\partial u}{\partial t} = \frac{\kappa\rho g\pi}{\mu}\frac{\partial}{\partial r}\left(r\frac{\partial u^2}{\partial r}\right). \tag{1.3}$$

If we nondimensionalize via

$$r^* = \frac{r}{R}, \quad t^* = \frac{Q}{m_+ R^4}\,t, \quad u = \frac{Q}{R^2}\,u^* \tag{1.4}$$

with R denoting the characteristic length scale in radial direction, and Q the volume initially contained in the liquid mound, and denote

$$\frac{m_+}{m_-} = 1 + \varepsilon \quad \text{with} \quad \varepsilon = \frac{\sigma_-}{\sigma_+ - \sigma_-}, \tag{1.5}$$

where

$$m_+ = \frac{2m\mu\sigma_+}{\kappa\rho\, g} \quad \text{and} \quad m_- = \frac{2m\mu(\sigma_+ - \sigma_-)}{\kappa\rho\, g}, \tag{1.6}$$

the nondimensional problem is, after dropping the $'*'$

$$\frac{\partial u}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u^2}{\partial r}\right), \qquad \text{for} \qquad \frac{\partial u}{\partial t} > 0, \tag{1.7a}$$

$$\frac{\partial u}{\partial t} = \frac{1+\varepsilon}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u^2}{\partial r}\right), \qquad \text{for} \qquad \frac{\partial u}{\partial t} < 0. \tag{1.7b}$$

Since the spreading mound $u(r,t)$ is radially symmetric the boundary condition at the symmetry axis is

$$\frac{\partial u}{\partial r} = 0 \quad \text{at} \quad r = 0. \tag{1.8}$$

As for the porous medium equation, solutions to the Cauchy problem for (1.7a–1.7b) have compact support

$$u = 0 \qquad \text{at} \qquad r = r_0(t) \tag{1.9}$$

and a bounding interface $r_0(t)$ with finite speed of propagation, see *e.g.* [5] and [6] for general reviews and [7] for this problem. Furthermore, continuity of u itself and $\partial u/\partial r$ across the interface $r_c(t)$, which is defined by $\partial u(r_c(t),t)/\partial t = 0$, is assumed; see *e.g.* [8].

Each of the Equations (1.7a) and (1.7b) is an example of the porous-medium equation and has similarity solutions of the form

$$u(r,t) = \frac{1}{\sqrt{t}}\, f(\eta;\varepsilon)\,, \quad \text{with} \quad \eta = r\, t^{-\frac{1}{4}}\,. \tag{1.10}$$

However, due to the differing diffusion coefficients, the respective solutions cannot be matched at the interface unless $\varepsilon = 0$. If $\varepsilon \neq 0$ the jump discontinuity of the coefficient in (1.7a) and (1.7b) across the free interface $r_c(t)$ renders the integral

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_0^{r_0(t)} r\, u(r,t)\,\mathrm{d}r \neq 0. \tag{1.11}$$

This means, the global condition, here conservation of the fluid mass, is not obeyed if $\varepsilon \neq 0$. Indeed, as we will see in more detail later, the very fact that this conservation law is violated prevents the *a priori* determination of the similarity index. On the other hand, we will solve this problem numerically and observe that the solution enters, for large times, a self-similar regime, which differs from (1.10); see also [8]. Similarity solutions of this type, *i.e.*, where the similarity index can not be obtained *a priori* by dimensional analysis, are termed second-kind similarity solutions.

The problem (1.7a, 1.7b), is a special case of the radially symmetric d-dimensional problem

$$\frac{\partial u}{\partial t} = \left[1 + \varepsilon H\left(-\frac{\partial u}{\partial t}\right)\right] \frac{1}{r^{d-1}} \frac{\partial}{\partial r}\left(r^{d-1} \frac{\partial u^{k+1}}{\partial r}\right), \quad r \geq 0$$

$$u(r,0) = F(r)\,, \qquad \int_0^{r_0} r^{d-1} F(r)\,\mathrm{d}r = 1\,. \tag{1.12}$$

in conjunction with the boundary conditions (1.8–1.9) and conditions at the interface, for $d = 2$ and $k = 1$, where H denotes the Heaviside operator.

It has been shown, see [7], that for $k > 0$, $d \geq 1$ the Cauchy problem has a unique solution in a class of compactly supported, non-negative, maximal viscosity solutions. Furthermore, they could show that, as $t \to \infty$, every maximal viscosity solution with compactly supported initial data converges to a similarity solution of the second kind.

Here, we will construct the second-kind similarity solution to (1.12) by making use of the violation of a conservation law to obtain a global condition relating the unknown similarity index to a certain value of the corresponding second-kind similarity profile. The asymptotic method presented here represents a generalization of those developed in [9] for Barenblatt's filtration equation, where we combine Lie-group and perturbation theory to derive the unknown similarity exponent. Here, we show how our method can be extended to nonlinear and higher dimensional PDE's.

Interestingly, we find that, while for the special case of $d = 2$, $k = 1$, our results agree with those by [10] who computed the similarity index by employing the inverse function theorem, but differ, even qualitatively, from those found in [11] where this problem was investigated using renormalization group methods.

We then consider the second-kind dipole solutions to (1.12), its application to the special case of the flood problem and numerically investigate the convergence of compactly supported, positive initial data to the second-kind similarity solutions.

2. The perturbation method

In [9] it has been shown how the second-kind similarity solution of Barenblatt's filtration equation are obtained if the problem is viewed as a perturbation of a corresponding problem with known similarity solution. More precisely, one can view the Lie group of the problems with second-kind similarity as a perturbation of the Lie group of a corresponding problem with known similarity solution. As a consequence of this one obtains the proper perturbation *ansatz* for such problems, *i.e.*, not only the functions but also the similarity exponents are functions of the perturbation parameter. In our case we make use of the Lie group of the porous medium equation; see [12]. Based on the method developed in [9] we start our analysis with

$$u(x,t)=t^{\alpha(\varepsilon)}f(\eta;\varepsilon) \qquad \eta=rt^{\beta(\varepsilon)}\,. \tag{2.1}$$

If we substitute this in (1.12), we obtain

$$\beta(\varepsilon)=-\frac{k\,\alpha(\varepsilon)+1}{2}\,, \tag{2.2}$$

and the similarity problem for $0\le\eta\le\eta_c$,

$$\alpha(\varepsilon)\eta^{d-1}f(\eta;\varepsilon)-\frac{k\alpha(\varepsilon)+1}{2}\eta^d\frac{\mathrm{d}f}{\mathrm{d}\eta}=(1+\varepsilon)\frac{\mathrm{d}}{\mathrm{d}\eta}\left(\eta^{d-1}\frac{\mathrm{d}f^{k+1}}{\mathrm{d}\eta}\right) \tag{2.3a}$$

and for $\eta_c\le\eta\le\eta_0$,

$$\alpha(\varepsilon)\eta^{d-1}f(\eta;\varepsilon)-\frac{k\alpha(\varepsilon)+1}{2}\eta^d\frac{\mathrm{d}f}{\mathrm{d}\eta}=\frac{\mathrm{d}}{\mathrm{d}\eta}\left(\eta^{d-1}\frac{\mathrm{d}f^{k+1}}{\mathrm{d}\eta}\right), \tag{2.3b}$$

where η_0 is the point where f vanishes and η_c, where the interface condition

$$\frac{\mathrm{d}}{\mathrm{d}\eta}\left(\eta^{d-1}\frac{\mathrm{d}f^{k+1}}{\mathrm{d}\eta}\right)=0 \quad \text{at } \eta=\eta_c \tag{2.4}$$

holds. The normalization takes on the form

$$\int_0^{\eta_0}\eta^{d-1}f(\eta;\varepsilon)\mathrm{d}\eta=1. \tag{2.5}$$

In order to determine the similarity index $\alpha(\varepsilon)$ one can make use of a conservation law for the case $\varepsilon=0$. If $\varepsilon\neq0$ this conservation law is violated. However, as we observe next, we obtain instead a generalized relation, we call *a dissipation law*. Since for our problem conservation of mass is violated we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^{r_0(t)}r^{d-1}u(r,t)\,\mathrm{d}r&=\int_0^{r_c}(1+\varepsilon)\frac{\partial}{\partial r}\left(r^{d-1}\frac{\partial u^{k+1}}{\partial r}\right)\mathrm{d}r\\&\quad+\int_{r_c}^{r_0}\frac{\partial}{\partial r}\left(r^{d-1}\frac{\partial u^{k+1}}{\partial r}\right)\mathrm{d}r=\varepsilon\,r_c^{d-1}\frac{\partial u^{k+1}}{\partial r}(r_c(t),t)\,.\end{aligned} \tag{2.6}$$

This condition in conjunction with the similarity form determines the similarity index $\alpha(\varepsilon)$ for $\varepsilon\neq0$. We observe that (2.6) becomes, in view of (2.1)

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(t^{\alpha(\varepsilon)-\beta(\varepsilon)d}\right)\int_0^{\eta_0}\eta^{d-1}f(\eta;\varepsilon)\,\mathrm{d}\eta=\varepsilon\,\eta_c^{d-1}\,t^{\alpha(\varepsilon)(k+1)-\beta(\varepsilon)(d-2)}\frac{\mathrm{d}f^{k+1}}{\mathrm{d}\eta}(\eta_c)\,.$$

Hence, in view of (2.5) we obtain the relation

$$\alpha(\varepsilon)+\frac{d}{k\,d+2}=\varepsilon\frac{2}{k\,d+2}\eta_c^{d-1}\frac{\mathrm{d}f^{k+1}}{\mathrm{d}\eta}(\eta_c)\ . \tag{2.7}$$

We assume now that $\varepsilon \ll 1$ and we make the perturbation *ansatz*

$$\alpha(\varepsilon)=\alpha_0+\varepsilon\alpha_1+\varepsilon^2\alpha_2+\cdots\ ,\quad \beta(\varepsilon)=-\frac{k\alpha_0+1}{2}-\varepsilon\frac{k\alpha_1}{2}-\varepsilon^2\frac{k\alpha_2}{2}-\cdots\ , \tag{2.8-2.9}$$

and

$$f(\eta;\varepsilon)=f_0(\eta)+\varepsilon f_1(\eta)+\varepsilon^2 f_2(\eta)+\cdots\ . \tag{2.10}$$

If we substitute (2.8–2.10) in Equation (1.12) (or (2.3a–2.3b)), we obtain a sequence of ordinary differential equations, together with the normalization

$$\int_0^{\eta_0}\eta^{d-1}\,f_0(\eta)\,\mathrm{d}\eta=1, \tag{2.11}$$

$$\int_0^{\eta_0}\eta^{d-1}\,f_i(\eta)\,\mathrm{d}\eta=0,\quad i=1,2,\ldots\ . \tag{2.12}$$

Furthermore, we obtain a sequence of equations from our relation for the similarity index, namely,

$$\alpha_0=-\frac{d}{k\,d+2}, \tag{2.13}$$

$$\alpha_1=\frac{2}{k\,d+2}\,\eta_c^{d-1}\frac{\mathrm{d}f_0^{k+1}}{\mathrm{d}\eta}(\eta_c)\,, \tag{2.14}$$

$$\alpha_2=\frac{2}{k\,d+2}\,\eta_c^{d-1}(k+1)\frac{\mathrm{d}}{\mathrm{d}\eta}\left(f_0^k f_1\right)\Big|_{\eta=\eta_c}\ . \tag{2.15}$$

$$\vdots$$

Hence we finally need to determine the values $f_i(\eta_c)$. To leading order we have the similarity problem for the porous medium equation which has the well-known Barenblatt-Pattle solution. Here we obtain

$$\alpha_0\eta^{d-1}\,f_0(\eta)-\frac{k\alpha_0+1}{2}\eta^d\frac{\mathrm{d}f_0}{\mathrm{d}\eta}=\frac{\mathrm{d}}{\mathrm{d}\eta}\left(\eta^{d-1}\frac{\mathrm{d}f_0^{k+1}}{\mathrm{d}\eta}\right),\quad 0\le\eta\le\eta_0\,,$$

which can be written as

$$\left(\frac{k\,d+2}{2}\alpha_0+\frac{d}{2}\right)\eta^{d-1}f_0(\eta)=\frac{\mathrm{d}}{\mathrm{d}\eta}\left(\eta^{d-1}\frac{\mathrm{d}f_0^{k+1}}{\mathrm{d}\eta}+\frac{k\alpha_0+1}{2}\eta^d\,f_0(\eta)\right).$$

This we integrate and recall that $\alpha_0=-d/(kd+2)$ to obtain

$$f_0(\eta)=\left[\frac{k}{2(k\,d+2)(k+1)}\left(\eta_0^2-\eta^2\right)\right]^{\frac{1}{k}}\,, \tag{2.16}$$

where η_0 is determined through the normalization condition

$$\int_0^{\eta_0}\eta^{d-1}\,f_0(\eta)\,\mathrm{d}\eta=\left(\frac{k}{2(k\,d+2)(k+1)}\right)^{\frac{1}{k}}\int_0^{\eta_0}\eta^{d-1}\,(\eta_0^2-\eta^2)^{\frac{1}{k}}\,\mathrm{d}\eta=1\,.$$

Thus

$$\eta_0^{\frac{kd+2}{k}} = 2\left(\frac{2(kd+2)(k+1)}{k}\right)^{\frac{1}{k}} \frac{1}{B\left(\frac{d}{2},\frac{k+1}{k}\right)}, \tag{2.17}$$

where B denotes the Beta function. Since the interface condition (2.4) yields

$$\eta_c = \sqrt{\frac{kd}{kd+2}}\,\eta_0\,, \tag{2.18}$$

we obtain for α_1 the formula

$$\alpha_1 = -\left(\frac{kd}{kd+2}\right)^{\frac{d}{2}}\left(\frac{2}{kd+2}\right)^{\frac{2k+1}{k}} \frac{1}{B\left(\frac{d}{2},\frac{k+1}{k}\right)}. \tag{2.19}$$

Next, we find the exponents α_i, $i=2,3,\ldots$. For this we now have only to solve the linear equations of second order for $f_{i-1}(\eta)$ in their respective interval of validity and require continuity across the interface at η_c, in order to find the value $f_{i-1}(\eta_c)$. We demonstrate this here for $i=2$. To $O(\varepsilon)$ we obtain from Equation (2.3a–2.3b) for $0\le\eta\le\eta_c$

$$\frac{\mathrm{d}}{\mathrm{d}\eta}\left[\eta^{d-1}(k+1)\frac{\mathrm{d}}{\mathrm{d}\eta}\left(f_0^k f_1\right)+\frac{1}{kd+2}\eta^d f_1\right] = \alpha_1\left(\eta^{d-1} f_0 - \frac{k}{2}\eta^d\frac{\mathrm{d}f_0}{\mathrm{d}\eta}\right) - \frac{\mathrm{d}}{\mathrm{d}\eta}\left(\eta^{d-1}\frac{\mathrm{d}f_0^{k+1}}{\mathrm{d}\eta}\right), \tag{2.20a}$$

and for $\eta_c\le\eta\le\eta_0$:

$$\frac{\mathrm{d}}{\mathrm{d}\eta}\left[\eta^{d-1}(k+1)\frac{\mathrm{d}}{\mathrm{d}\eta}\left(f_0^k f_1\right)+\frac{1}{kd+2}\eta^d f_1\right] = \alpha_1\left(\eta^{d-1} f_0 - \frac{k}{2}\eta^d\frac{\mathrm{d}f_0}{\mathrm{d}\eta}\right). \tag{2.20b}$$

We can now integrate (2.20a) from 0 to η and (2.20b) from η to η_0 to obtain for $0\le\eta\le\eta_c$:

$$\eta^{d-1}(k+1)\frac{\mathrm{d}}{\mathrm{d}\eta}\left(f_0^k f_1\right)+\frac{1}{kd+2}\eta^d f_1 = \alpha_1\int_0^{\eta}\eta^{d-1} f_0 - \frac{k}{2}\eta^d\frac{\mathrm{d}f_0}{\mathrm{d}\eta}\,\mathrm{d}\eta - \eta^{d-1}\frac{\mathrm{d}f_0^{k+1}}{\mathrm{d}\eta}, \tag{2.21a}$$

and for $\eta_c\le\eta\le\eta_0$:

$$\eta^{d-1}(k+1)\frac{\mathrm{d}}{\mathrm{d}\eta}\left(f_0^k f_1\right)+\frac{1}{kd+2}\eta^d f_1 = \alpha_1\int_{\eta}^{\eta_0}\eta^{d-1} f_0 - \frac{k}{2}\eta^d\frac{\mathrm{d}f_0}{\mathrm{d}\eta}\,\mathrm{d}\eta. \tag{2.21b}$$

Note, that now, by evaluating (2.21b) at η_c we can determine α_2 from

$$\alpha_2 = -\frac{2}{kd+2}\left(\alpha_1 I_0(\eta_c) + \frac{1}{kd+2}\eta_c^d f_1(\eta_c)\right), \tag{2.22}$$

where

$$I_0(\eta) = \left(\frac{k\eta_0^2}{2(kd+2)(k+1)}\right)^{\frac{1}{k}}\int_{\eta}^{\eta_0}\lambda^{d-1}\left(\frac{\eta_0^2}{\eta_0^2-\lambda^2}\right)^{\frac{k-1}{k}}\mathrm{d}\lambda \tag{2.23}$$

and

$$f_1(\eta_c) = \frac{8}{\eta_0^d B\left(\frac{d}{2},\frac{k+1}{k}\right)}\left(\alpha_1\left[I_7(\eta_c)I_2(\eta_c) - \eta_0^{\frac{2(k-1)}{k}}\left(I_5(\eta_c)-I_6(\eta_c)\right)\right]\right.$$
$$\left. + \eta_0^{\frac{2(k-1)}{k}} I_4(\eta_c) + \frac{k\mathrm{d}\eta_0^2}{2(kd+2)^2}\left(\frac{k}{2(kd+2)(k+1)}\right)^{\frac{1}{k}} I_7(\eta_c)\right). \tag{2.24}$$

The derivation of (2.24) is given in Appendix B.

In the following section we will apply our results to the problem of the spreading ground-water mound in a porous medium, we expounded in the introduction. This will be followed by a comparison with our numerical solution to (2.3a–2.3b).

3. Spreading of a ground-water mound

For the problem of the spreading ground-water mound, where $k=1$ and $d=2$ our results come out in a particularly simple form. We have

$$\alpha_0=-\frac{1}{2}, \qquad \eta_0^2=8, \qquad \eta_c=1, \tag{3.1}$$

$$f_0=\begin{cases} \frac{1}{16}\left(8-\eta^2\right) & \eta\le\sqrt{8} \\ 0 & \eta>\sqrt{8} \end{cases},$$

and

$$\alpha_1=-\frac{1}{8}. \tag{3.2}$$

For the integrals we need for $f_1(\eta_c)$ and α_2 we obtain:

$$I_0(\eta)=\frac{1}{4}\left(8-\eta^2\right), \qquad I_1(\eta)=\frac{1}{4}\eta^2, \qquad I_2(\eta)=\frac{1}{\sqrt{8}}\log\left(\frac{\sqrt{8}+\eta}{\sqrt{8}-\eta}\right),$$

$$I_3(\eta)=-\frac{\eta^2}{128}, \quad I_4(\eta)=-\frac{\eta^4}{512}, \quad I_7(\eta)=\frac{\eta^2}{2}, \quad I_8(\eta)=\frac{1}{8}\left(8-\eta^2\right),$$

and

$$I_5(\eta)=-\frac{1}{16}\left[8\log 8+\eta^2(\log 8-1)-\left(8-\eta^2\right)\log\left(8-\eta^2\right)\right],$$
$$I_6(\eta)=\frac{1}{16}\left[8(\log 8+\log 4)-\eta^2\log(\eta^2)-\left(8-\eta^2\right)\log\left(8-\eta^2\right)-\eta^2\log 4\right].$$

From these explicit expressions in (2.22) one can easily calculate

$$\alpha_2=0{\cdot}05539339761. \tag{3.3}$$

This agrees with the result in [10]. For this problem there is also another result in [11] available, who employ a renormalization group method. However, while their value for α_1 agrees with ours, their value for α_2 does not. In the next paragraph we compare the results of [11] and ours to numerical results.

3.1. Comparison to numerical solution

At this point, it is useful to numerically solve the ground-water mound problem. For this purpose, we rewrite the Equation (2.3a) and (2.3b), for $k=1$ and $d=2$, in the following form:

$$f_{\eta\eta}=\frac{\alpha}{2(1+\varepsilon)}-\frac{f_\eta}{\eta}-\frac{f_\eta}{f}\left[f_\eta+\frac{1+\alpha}{4(1+\varepsilon)}\right] \quad \text{for } 0<\eta<\eta_c, \tag{3.4a}$$
$$f_{\eta\eta}=\frac{\alpha}{2}-\frac{f_\eta}{\eta}-\frac{f_\eta}{f}\left[f_\eta+\frac{1+\alpha}{4}\right] \quad \text{for } \eta_c<\eta<1, \tag{3.4b}$$

with the boundary condition at $\eta = 0$,

$$f_\eta(0) = 0\,, \tag{3.5}$$

and the interface and continuity conditions at $\eta = \eta_c$,

$$\alpha f(\eta_c) - \frac{1+\alpha}{2}\eta_c f_\eta(\eta_c) = 0, \quad [f]_{\eta_c^-}^{\eta_c^+} = 0, \quad [f_\eta]_{\eta_c^-}^{\eta_c^+} = 0\,. \tag{3.6-3.8}$$

For the boundary condition at the interface to the porous medium, we normalize without loss of generality $\eta_0 = 1$, such that

$$f(1) = 0, \tag{3.9}$$

Finally, we find it numerically more convenient to replace the integral condition, or the dissipation law, by an extra boundary condition at the liquid/porous-medium interface $\eta = 1$. There, we require no flux across this boundary. Hence, from (2.3b) this results in the condition

$$f_\eta(1) = -\frac{1+\alpha}{4}, \tag{3.10}$$

The problem (3.4a–3.10) is essentially a two-point boundary-value problem for a second-order differential equation, but with three instead of two boundary conditions at $\eta = 0$ and 1. The extra boundary condition fixes the value of $\alpha(\varepsilon)$.

We solve this problem numerically via a shooting method. For this purpose, we convert the second-order ODE into a system of first-order ODEs using the settings $y_1(\eta) := f(\eta)$ and $y_2(\eta) := f_\eta(\eta)$, which is then solved using an explicit Adams-scheme implemented in the LSODE-package [13]. The code incorporates a local error estimator for $y_i(\eta)\, i = 1, 2$, and automatically adapts its step-size so that the estimated error is within a given tolerance for the relative error.

The integration is first carried out for (3.4b) up to η_c, starting from the right end point. In a second step, the integration is continued to the left with the $f(\eta_c)$ and $f_\eta(\eta_c)$ obtained from the previous run, now using (3.4a). Note that η_c has to be determined as part of the first step; since preliminary runs indicated that f was monotone at η_c, this can be done very easily through bisection. Near the left end point, f will in general not fulfill (3.5); rather, this requirement must be fulfilled in order to determine the similarity exponent $\alpha(\varepsilon)$. It turns out that, near $\eta = 0$, f_η depends monotonically on α, so a bisection method can again be used. In both bisection schemes, we started with a rather generous choice for the bracketing interval, making sure that the value of interest was included, then calculated the value of $f(\eta_c)$, for example, and replaced one of the points of the interval, according to the sign of $f(\eta_c)$. This procedure was repeated until both the length of the interval and $f(\eta_c)$ had dropped beneath prescribed tolerances $\triangle\eta_c$ and $\triangle f$. Similarly for $\alpha(\varepsilon)$ and f_η near 0, with tolerances $\triangle\alpha$ and $\triangle f_\eta$ for the length of the bracketing interval and for f_η near zero.

Special attention is required when integrating the ODE near $\eta = 0$ and $\eta = 1$, where η and f vanish, respectively, since these quantities appear in the denominator of certain terms of (3.4a) and (3.4b). We avoid these regions by starting the integration at an η_r slightly smaller than 1, and using a linear approximation to obtain a good choice for $f(\eta_r)$,

$$f(\eta_r) \approx f(1) - \frac{1+\alpha}{4}(\eta_r - 1) = \frac{1+\alpha}{4}(1 - \eta_r) > 0.$$

Likewise, we do not integrate up to zero, but use a small but positive value for the left end point η_l instead.

The numerical trials were carried out using the standard choice of tolerances and truncation parameters, $\mathrm{tol} = 10^{-10}$, $\eta_l = 10^{-4}$, $\eta_r = 1 - 10^{-5}$, $\triangle\alpha = 10^{-11}$, $\triangle f_\eta = 10^{-8}$, $\triangle\eta_c = 10^{-11}$, $\triangle f = 10^{-7}$.

The results for $\alpha(\varepsilon)$ of the numerical computations for a standard choice of tolerances and truncation parameters plus convergence checks for the truncation parameters η_l and η_r are shown in Table 1 in Appendix A. They show very good agreement with the analytical value $\alpha(0) = -0{\cdot}5$ for the first-kind similarity case.

We now compare our results with the asymptotic theory. To this end, we calculate, for each ε, the values

$$\alpha_1 = \frac{\alpha(\varepsilon) + 0{\cdot}5}{\varepsilon}, \quad \alpha_2 = \frac{\alpha(\varepsilon) + 0{\cdot}5 + 0{\cdot}125\varepsilon}{\varepsilon^2}.$$

As ε approaches 0, these values should converge to the theoretical predictions. The results are shown in Table 2 in Appendix A. Convergence can indeed be observed for α_1, and for α_2. For the latter, the numerical error prevents α_2 to get closer to the theoretical value for $\varepsilon < 0{\cdot}005$.

The numerical estimates can be improved by extrapolation of the tabulated values for $\alpha(\varepsilon)$. To avoid the influence of numerical errors from the inclusion of $\alpha(\varepsilon)$ for very small ε, we only used the values of Table 1 for $\varepsilon \geq 0{\cdot}01$ to compute the extrapolation polynomial, and read off the following values for the lower-order coefficients,

$$\alpha_0 = -0{\cdot}5000000064, \quad \alpha_1 = -0{\cdot}1249995996, \quad \alpha_2 = 0{\cdot}05537538810.$$

When we compare the numerical solution for the decay rate $\alpha(\varepsilon)$ with our asymptotic results we observe a significant improvement from our $O(\varepsilon^2)$ result; see Figure 2. Here, we compare our results with those from [11]. To that end we show, as they did in their article, the quantity $10(-\alpha(\varepsilon) - 1/2)$ as a function of ε. We observe in Figure 3, that their result is also qualitatively very different from our results. In view of the discrepancy of the earlier results on α_2 for Barenblatt's filtration equation in [14] and [9] we are not sure if the reason might be a problem with the way the renormalization group method is applied, since a derivation of their higher order exponents was, unfortunately, not given.

Finally we like to address the question of convergence of solutions of (1.12) (with $k = 1$, $d = 2$) of compactly supported positive initial data $F(r)$ to a similarity solution. For the numerical integration of (1.12) we make use of the IMSL routine DMOLCH. This routine uses the method of lines, where the spatial discretization is achieved by collocation using cubic Hermite polynomials. The routine assumes that the initial data satisfy the boundary conditions and have smooth derivatives. In all our calculations we let the number of Hermite knots

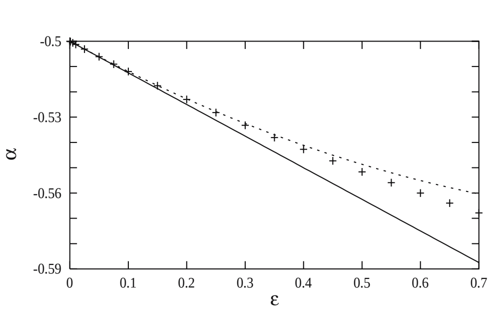

Figure 2. Numerical result (+). First order (—) and second order (- - -) asymptotic results.

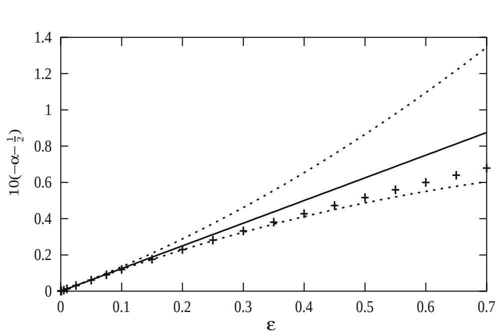

Figure 3. Numerical results (+), the first-order asymptotic result (—), Goldenfeld *et al.* second-order asymptotic result (- - -), our second-order asymptotic result (- · -).

$N=500$ and specify the error tolerance tol$=10^{-7}$. If we now multiply the solution $u(r,t)$ by $t^{-\alpha(\varepsilon)}$ and r by $t^{-\beta(\varepsilon)}$, then, in these scales, we observe, that the solution tends to a stationary limit as $t\to\infty$; see also [15]. As an example we set $\varepsilon=0{\cdot}3$ and use our asymptotic results (3.1–3.3) in (2.8-2.9). For the initial condition we use the function

$$F(r)=(1-r^2)^3 \quad \text{for } -1<r<1\,, \tag{3.11}$$

and zero otherwise. We observe in Figure 4, that already for $t=10$ the solution is, within graphical resolution, stationary, *i.e.*, in self-similar form.

4. The dipole problem

We demonstrate in this section that our method also extends to problems with different underlying conservation laws. Here, we consider instead of conservation of mass for the underlying porous-medium equation ((1.12) for $\varepsilon=0$), the conservation of the flux and for simplicity restrict ourselves to the case where $d=1$. We therefore assume now, that the resulting problem is not symmetric about $x=0$ and impose Dirichlet boundary conditions at $x=0$. We notice first that this problem does not admit an energy integral; however, if we consider the first moment, we immediately see, after integrating parts, that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^\infty x\,u(x,t)\mathrm{d}x=u^{k+1}(0,t)\,.$$

Thus, the flux is conserved if Dirichlet boundary conditions are obeyed at $x=0$. This problem has similarity solutions of the first kind, the so-called dipole solutions, describing the large-time behavior of solutions with initial data on the halfline, [16]. Existence of a unique continuous weak solution with compact support, as well as convergence to the dipole solution in the large-time limit has been shown in [17,18].

One typical application of this problem for $k=1$, concerns the impact of a flood on the motion of groundwater. If for example at a certain time the level of a liquid begins to rise quickly at the symmetry-axis $x=0$ of a porous layer and after a short duration is again withdrawn, the large-time behavior of the liquid distribution in the porous medium shows a front moving with finite velocity further into the porous medium, while at the boundary $x=0$ fluid is lost at a constant rate.

Here, we are concerned with the effect of some additional loss of liquid in the pores of the layer, which, of course, to a certain extent, is the case for all materials.

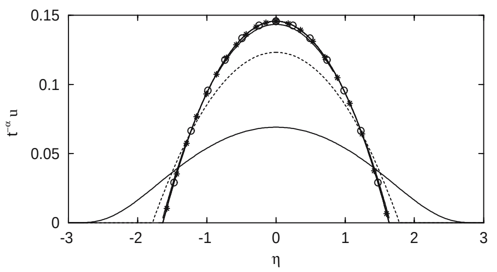

Figure 4. $t^{-\alpha}u(x,t)$ in similarity variables at times $t=$ 0·01 (—), 0·1 (···), 1 (- - -), 10 ($-\circ-$), 100 ($-*-$) for $\varepsilon=0{\cdot}3$.

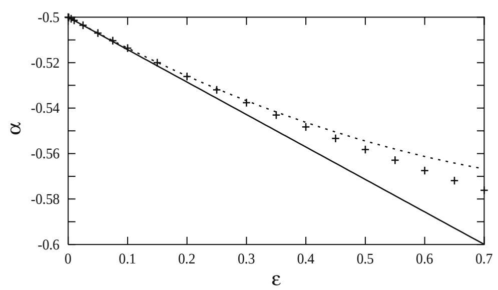

Figure 5. Numerical result (+), first-order (—) and second-order (- - -) asymptotic results.

When considering an analogous situation as for the problem of the groundwater mound ($k=1$), where the fluid spreads in a porous medium, we will observe that also conservation of the flux is violated and replaced by a corresponding *dissipation law*.

Hence, when looking for the large-time behavior of solutions to the problem

$$\frac{\partial u}{\partial t}=\left[1+\varepsilon H\left(-\frac{\partial u}{\partial t}\right)\right]\frac{\partial^2 u^{k+1}}{\partial x^2}, \qquad x\geq 0$$
$$u(x,0)=F(x)\,, \quad \int_0^{x_0} xF(x)\mathrm{d}x=1\,, \tag{4.1}$$

we expect the similarity solutions to be of second kind. As in the previous section we construct them by using the composite-expansion *ansatz* (2.1) in (4.1) to obtain $\beta(\varepsilon)=-[k\alpha(\varepsilon)+1]/2$ and the similarity problem for $0\leq\eta\leq\eta_c$:

$$\alpha(\varepsilon)\eta\, f(\eta;\varepsilon)-\frac{k\alpha(\varepsilon)+1}{2}\eta^2\frac{\mathrm{d}f}{\mathrm{d}\eta}=(1+\varepsilon)\,\frac{\mathrm{d}}{\mathrm{d}\eta}\left(\eta\frac{\mathrm{d}f^{k+1}}{\mathrm{d}\eta}-f^{k+1}(\eta;\varepsilon)\right)\,, \tag{4.2a}$$

and for $\eta_c\leq\eta\leq\eta_0$:

$$\alpha(\varepsilon)\eta\, f(\eta;\varepsilon)-\frac{k\alpha(\varepsilon)+1}{2}\eta^2\frac{\mathrm{d}f}{\mathrm{d}\eta}=\frac{\mathrm{d}}{\mathrm{d}\eta}\left(\eta\frac{\mathrm{d}f^{k+1}}{\mathrm{d}\eta}-f^{k+1}(\eta;\varepsilon)\right). \tag{4.2b}$$

In this problem $f(\eta;\varepsilon)$ vanishes at $\eta=0$ and $\eta=\eta_0$. Here η_c is the point where the interface condition

$$\frac{\mathrm{d}^2 f^{k+1}}{\mathrm{d}\eta^2}=0 \quad \text{at} \quad \eta=\eta_c \tag{4.3}$$

holds and the normalization is here

$$\int_0^{\eta_0}\eta\, f(\eta;\varepsilon)\mathrm{d}\eta=1. \tag{4.4}$$

The *dissipation law* for the mass-flux takes on the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^{x_0(t)} xu(x,t)\mathrm{d}x=\int_0^{x_c}(1+\varepsilon)\,\frac{\partial}{\partial x}\left(x\frac{\partial u^{k+1}}{\partial x}-u^{k+1}\right)\mathrm{d}x$$
$$+\int_{x_c}^{x_0}\frac{\partial}{\partial x}\left(x\frac{\partial u^{k+1}}{\partial x}-u^{k+1}\right)\mathrm{d}x=\varepsilon\left(x_c\frac{\partial u^{k+1}}{\partial x}\,(x_c(t),t)-u^{k+1}\,(x_c(t),t)\right), \tag{4.5}$$

which, in conjunction with the similarity form, yields the following formula for the similarity index $\alpha(\varepsilon)$:

$$\alpha(\varepsilon)(k+1)+1=\varepsilon\left(\eta_c\frac{\mathrm{d}f^{k+1}}{\mathrm{d}\eta}\,(\eta_c)-f^{k+1}\,(\eta_c)\right)\,. \tag{4.6}$$

We assume now that $\varepsilon\ll 1$ and we make the perturbation *ansatz* (2.8-2.9–2.10) from which we obtain

$$\int_0^{\eta_0}\eta\, f_0(\eta)\mathrm{d}\eta=1, \tag{4.7}$$

$$\int_0^{\eta_0}\eta\, f_i(\eta)\mathrm{d}\eta=0, \qquad i=1,2,\ldots\,, \tag{4.8}$$

and

$$\alpha_0 = -\frac{1}{k+1}, \tag{4.9}$$

$$\alpha_1 = -\frac{1}{k+1}\left(f_0^{k+1}(\eta_c) - \eta_c \frac{\mathrm{d} f_0^{k+1}}{\mathrm{d}\eta}(\eta_c)\right), \tag{4.10}$$

$$\alpha_2 = -\left(f_0^k f_1 - \eta \frac{\mathrm{d}}{\mathrm{d}\eta}\left(f_0^k f_1\right)\right)|_{\eta=\eta_c}. \tag{4.11}$$

$$\vdots$$

In order to determine $f_0(\eta_c)$ we integrate the leading-order problem and obtain, in view of (4.9)

$$\eta \frac{\mathrm{d} f_0^{k+1}}{\mathrm{d}\eta} - f_0^{k+1}(\eta) + \frac{1}{2(k+1)}\eta^2 f_0(\eta) = 0. \tag{4.12}$$

The use of the integrating factor $[\eta^{(2k+1)/(k+1)} f_0]^{-1}$ then enables us to solve for f_0 (see also [16]):

$$f_0(\eta) = \left[\frac{k}{2(k+2)(k+1)}\left(\eta_0^{\frac{k+2}{k+1}} - \eta^{\frac{k+2}{k+1}}\right)\eta^{\frac{k}{k+1}}\right]^{\frac{1}{k}}, \tag{4.13}$$

and the normalization condition yields η_0:

$$\eta_0^{\frac{2(k+1)}{k}} = \frac{k+2}{k+1}\left(\frac{2(k+2)(k+1)}{k}\right)^{\frac{1}{k}} \frac{1}{B\left(\frac{k+1}{k+2}+1, \frac{k+1}{k}\right)}. \tag{4.14}$$

Since the interface condition (4.3) yields

$$\eta_c = \left(\frac{k(2k+3)}{2(k+1)^2}\right)^{\frac{k+1}{k+2}} \eta_0, \tag{4.15}$$

we obtain for α_1 the formula

$$\alpha_1(k) = -\frac{1}{k+1}\left(\frac{k(2k+3)}{2(k+1)^2}\right)^{\frac{2k+3}{k+2}}\left(\frac{k+2}{2(k+1)^2}\right)^{\frac{k+1}{k}} \cdot B^{-1}\left(\frac{k+1}{k+2}+1, \frac{k+1}{k}\right). \tag{4.16}$$

The index α_2 is again determined by solving the $O(\varepsilon)$ problem for $0 \le \eta \le \eta_c$, viz.

$$\frac{\mathrm{d}}{\mathrm{d}\eta}\left[(k+1)\left(\eta\frac{\mathrm{d}}{\mathrm{d}\eta}\left(f_0^k f_1\right) - f_0^k f_1\right) + \frac{\eta^2 f_1}{2(k+1)}\right] = \alpha_1\left(\eta f_0 - \frac{k}{2}\eta^2\frac{\mathrm{d} f_0}{\mathrm{d}\eta}\right) - \frac{\mathrm{d}}{\mathrm{d}\eta}\left(\eta\frac{\mathrm{d} f_0^{k+1}}{\mathrm{d}\eta} - f_0^{k+1}\right), \tag{4.17a}$$

and for $\eta_c \le \eta \le \eta_0$:

$$\frac{\mathrm{d}}{\mathrm{d}\eta}\left[(k+1)\left(\eta\frac{\mathrm{d}}{\mathrm{d}\eta}\left(f_0^k f_1\right) - f_0^k f_1\right) + \frac{\eta^2 f_1}{2(k+1)}\right] = \alpha_1\left(\eta f_0 - \frac{k}{2}\eta^2\frac{\mathrm{d} f_0}{\mathrm{d}\eta}\right). \tag{4.17b}$$

To solve this, we integrate (4.17a) from 0 to η and (4.17b) from η to η_0 and obtain for $0 \le \eta \le \eta_c$:

$$(k+1)\left(\eta\frac{\mathrm{d}}{\mathrm{d}\eta}(f_0^k f_1) - f_0^k f_1\right) + \frac{\eta^2 f_1}{2(k+1)} = \alpha_1 \int_0^{\eta} \lambda f_0 - \frac{k}{2}\lambda^2 \frac{\mathrm{d}f_0}{\mathrm{d}\lambda}\,\mathrm{d}\lambda - \eta\frac{\mathrm{d}f_0^{k+1}}{\mathrm{d}\eta} + f_0^{k+1} \quad (4.18a)$$

and for $\eta_c \le \eta \le \eta_0$:

$$(k+1)\left(\eta\frac{\mathrm{d}}{\mathrm{d}\eta}(f_0^k f_1) - f_0^k f_1\right) + \frac{\eta^2 f_1}{2(k+1)} = -\alpha_1 \int_{\eta}^{\eta_0} \lambda f_0 - \frac{k}{2}\lambda^2 \frac{\mathrm{d}f_0}{\mathrm{d}\lambda}\,\mathrm{d}\lambda. \quad (4.18b)$$

Thus, by (4.18b) we can determine α_2 from

$$\alpha_2 = -\frac{1}{k+1}\left(\alpha_1 J_1(\eta_c) + \frac{1}{2(k+1)}\eta_c^2 f_1(\eta_c)\right), \quad (4.19)$$

where

$$J_1(\eta) = \int_{\eta}^{\eta_0} \lambda f_0 - \frac{k}{2}\lambda^2 \frac{\mathrm{d}f_0}{\mathrm{d}\lambda}\,\mathrm{d}\lambda. \quad (4.20)$$

We solve the next-order problem to determine $f_1(\eta_c)$ in Appendix C.

5. The flood problem: asymptotic and numerical results

Let us look again at the case, where $k=1$ and recall the problem of the impact of a flood on the groundwater motion, briefly described in the last section. From our asymptotic results, we find in this case

$$\alpha_0 = -\frac{1}{2}, \quad \alpha_1 = -0{\cdot}1427743036, \quad \alpha_2 = 0{\cdot}06773941887. \quad (5.1)$$

For details see Appendix C.1.

For a comparison of our results with the numerical solution of this problem we recast the equations in the following form. For $0 < \eta < \eta_c$:

$$f_{\eta\eta} = -\frac{f_\eta^2}{f} + \frac{1}{2(1+\varepsilon)}\left[\alpha - \frac{1+\alpha}{2}\eta\frac{f_\eta}{f}\right], \quad (5.2a)$$

and for $\eta_c < \eta < 1$:

$$f_{\eta\eta} = -\frac{f_\eta^2}{f} + \frac{1}{2}\left[\alpha - \frac{1+\alpha}{2}\eta\frac{f_\eta}{f}\right] \quad (5.2b)$$

with the boundary and interface conditions at 0, η_c and 1,

$$f(0) = 0, \quad (5.3a)$$

$$[f]_{\eta_c^-}^{\eta_c^+} = 0, \qquad [f_\eta]_{\eta_c^-}^{\eta_c^+} = 0, \qquad \alpha f(\eta_c) - \frac{1+\alpha}{2}\eta_c f_\eta(\eta_c) = 0, \quad (5.3b)$$

$$f(1) = 0, \qquad f_\eta(1) = -\frac{1+\alpha}{4}, \quad (5.3c)$$

where we have normalized η_0 to 1.

Again, we have to solve a two-point boundary-value problem with one boundary condition more than the order of the differential equation, which is necessary to fix the unknown similarity exponent $\alpha(\varepsilon)$. The numerical algorithm follows a pattern very similar to the method used for the groundwater mound problem with comparable accuracy. Comparing the results with our asymptotic solution shows very good agreement as seen in Figure 5.

In Figure 6 we show some typical self-similar shapes for the dipole solution for various ε. We observe, that when the residual saturation σ_- approaches σ_+, *i.e.*, $1/(1+\varepsilon) \to 0$, the similarity shapes become symmetrical about their maximum. In this limit, the spreading rate $\beta(\varepsilon) = -[k\alpha(\varepsilon)+1]/2$ vanishes. This can be seen in Figure 6, which shows a range of spreading rates.

6. Calculation of waiting times

Finally, we like to make a few remarks on how solutions of (4.1) (with $k=1$) converge to the similarity solutions for various ε, for compactly supported positive initial data $F(x)$. For the numerical integration of (4.1) we use again the IMSL routine DMOLCH. Again, we let in all our calculations the number of Hermite knots $N=500$ and specify the error tolerance tol $=10^{-7}$. For the initial condition we use the function

$$F(x) = x(2-x)^3 \quad \text{for } 0 \le x < 2, \tag{6.1}$$

and zero otherwise.

At first we consider the case $\varepsilon = 0$. We observe in Figure 7 that the similarity solution emerges rather quickly. Furthermore, we note that, until about $t^* \simeq 0{\cdot}11$, the support of the initial data does not move, *i.e.*, we have a positive waiting time t^*. We illustrate these properties in the left side of Figure 8, where we compare the numerical solution to (4.1) with the solution to (5.2a–5.3c). Here, we scale the solution $f(\eta)$ such that its maximum agrees with the maximum of $u(x,t)$ and multiply $\eta_0 = 1$ by the right boundary of the support of $u(x,t)$, at a given time.

From the previous sections we know that, for $\varepsilon > 0$, the decay rate $\alpha(\varepsilon)$ increases, while the spreading rate $\beta(\varepsilon)$ decreases. Starting with the same initial data as above (6.1), we observe this behavior in Figure 7 (on the right) for $\varepsilon = 10$.

Furthermore, we find that the waiting time for $\varepsilon > 0$ is larger than for $\varepsilon = 0$. This has the effect that in such cases, the solution is almost in self-similar form before the boundary of its support begins to move, as can be observed on the right-hand side of Figure 8. Hence, interestingly, the location of the boundary is here completely described by the similarity solution.

Note however, that waiting times result to a certain extent from local effects [19]. For initial shapes other than (6.1), for example when the right boundary of the support is approached linearly, we observe zero waiting times. A classification for initial conditions with positive waiting time for the modified dipole problem, as well as the modified porous-medium equation is still not completely understood.

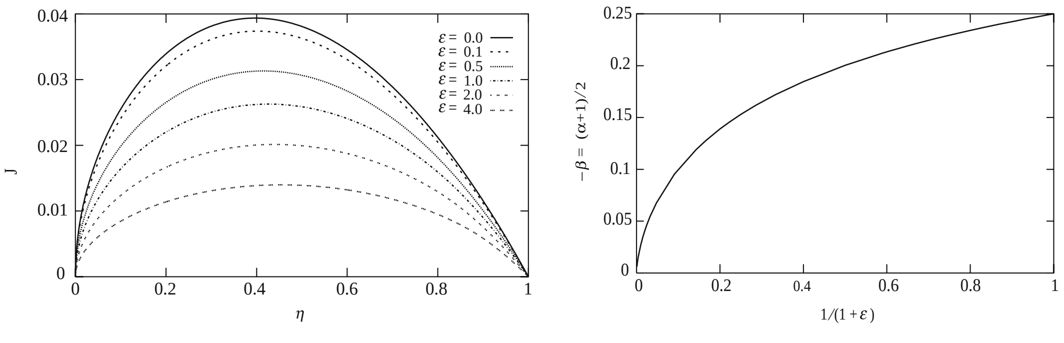

Figure 6. Self-similar dipole solutions for various ε (left). The spreading rate $-\beta$ as a function of ε (right).

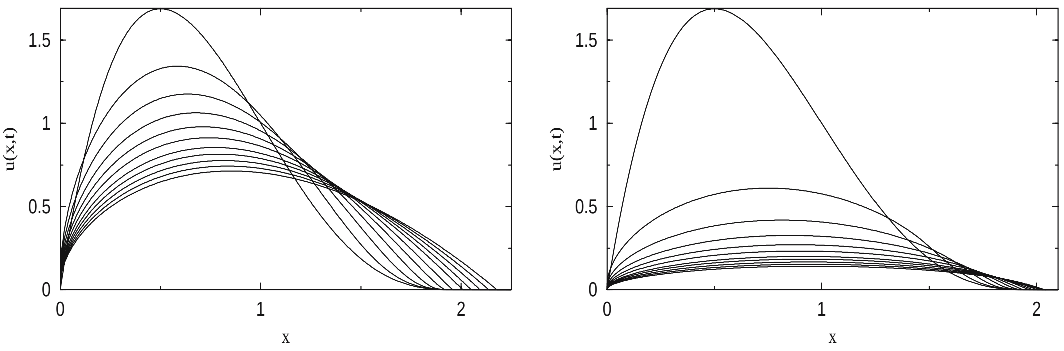

Figure 7. $u(x, t)$ for $t = 0, \ldots, 0{\cdot}2$ and for $\varepsilon = 0$ (left) and $\varepsilon = 10$ (right).

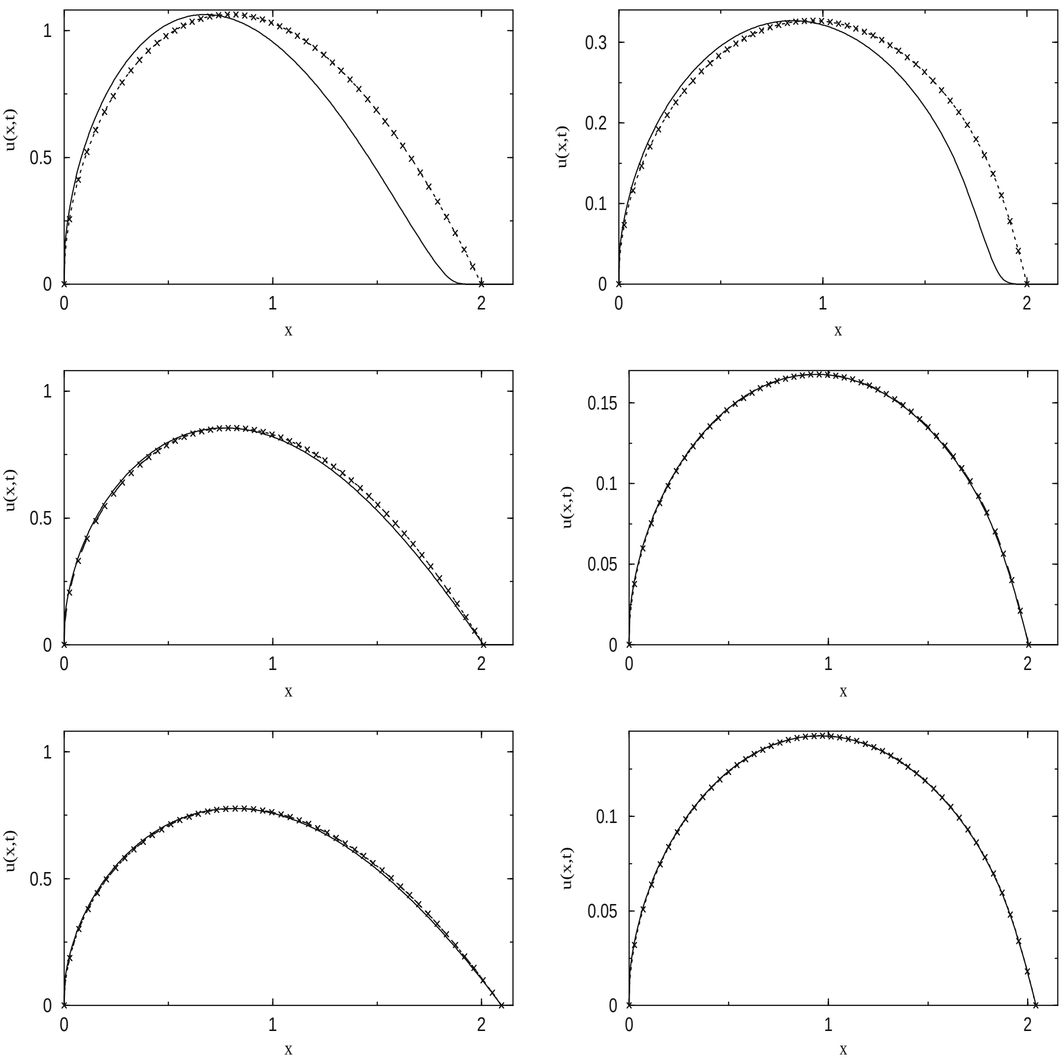

Figure 8. Comparison of $u(x, t)$ and the corresponding similarity solution $(\cdots \times \cdots)$ at $t = 0{\cdot}06, 0{\cdot}12, 0{\cdot}16$ for $\varepsilon = 0$ (left) and at $t = 0{\cdot}06, 0{\cdot}16, 0{\cdot}2$ for $\varepsilon = 10$ (right).

7. Conclusions

In this paper we derived the second-kind similarity solutions for the modified porous-medium equation using an approach that combines Lie-group and perturbation theory. We demonstrated how, for the special case of a spreading two-dimensional liquid mound, it is straightforward to determine the second-kind similarity exponent. Our results disagree with those of [11] to higher-order terms in the perturbation expansion of α. We also studied the second-kind dipole solutions. These similarity solutions occur for the Dirichlet initial-boundary-value problem and our method exploits the deviation from conservation of mass flux. We compared our results to numerical computations and showed how, for positive compactly supported initial data, the similarity solutions are approached.

We like to note that our method is not limited to the family of problems investigated in this paper. In general, the small perturbation parameter ε, needed to carry out the asymptotic expansion, is identified as the deviation of the second-kind process from a corresponding first-kind self-similar process, and the Lie group of the second-kind self-similar process emerges as a perturbation of the Lie group of the corresponding first-kind process. More generally, our method is applicable to problems in which at least one value of the similarity index, together with the similarity solution, first or second kind, is known. About this value the Lie group can then be perturbed and solutions for the related problems can be determined.

Acknowledgements

The author is most grateful to the comments of the anonymous referees.

Appendix A. Tables

Table 1. $\alpha(\varepsilon)$ for various $1-\eta_r$, η_l for a standard choice of tolerances and truncation parameters, tol $=10^{-10}$, $\eta_l=10^{-4}$, $\eta_r=1-10^{-5}$, $\triangle\alpha=10^{-11}$, $\triangle f_\eta=10^{-8}$, $\triangle\eta_c=10^{-11}$, $\triangle f=10^{-7}$.

ε	$1-\eta_r=10^{-5}$	$1-\eta_r=10^{-4}$	$1-\eta_r=10^{-3}$	$\eta_l=10^{-3}$	$\eta_l=10^{-2}$
0	−0·500000005	−0·500000014	−0·500001003	−0·500000499	−0·500049962
0·001	−0·500124949	−0·500124959	−0·500125948	−0·500125444	−0·500174898
0·005	−0·500623623	−0·500623633	−0·500624626	−0·500624118	−0·500673534
0·01	−0·501244496	−0·501244506	−0·501245504	−0·501244990	−0·501294359
0·025	−0·503090869	−0·503090879	−0·503091893	−0·503091362	−0·503140590
0·05	−0·506115352	−0·506115362	−0·506116402	−0·506115842	−0·506164840
0·075	−0·509076148	−0·509076158	−0·509077225	−0·509076636	−0·509125409
0·1	−0·511975789	−0·511975800	−0·511976893	−0·511976275	−0·512024827

Underlining indicates the digits which coincide with the values obtained for our standard choice.

Table 2. $\alpha_1(\varepsilon)$ and $\alpha_2(\varepsilon)$ computed from the numerical data, for various ε.

ε	α_1	α_2
0·001	−0·12495	0·05100
0·005	−0·1247	0·05508
0·01	−0·1244	0·05504
0·025	−0·1236	0·05461
0·05	−0·1223	0·05386
0·075	−0·1210	0·05313
0·1	−0·1198	0·05242

Appendix B. Derivation of $f_1(\eta_c)$

In order to determine $f_1(\eta_c)$ we multiply (2.21a–2.21b) by the factor $\eta^{1-d}\left(\eta_0^2-\eta^2\right)^{-1/k}$ and integrate the resulting equation for the interval $0\le\eta\le\eta_c$ from 0 to η, and the equation for the interval $\eta_c\le\eta\le\eta_0$ from η_c to η. This yields for $0\le\eta\le\eta_c$:

$$f_1(\eta)=\left(\frac{\eta_0^2}{\eta_0^2-\eta^2}\right)^{\frac{k-1}{k}} f_1(0)+\frac{2(kd+2)}{k\left(\eta_0^2-\eta^2\right)^{\frac{k-1}{k}}}\left[\alpha_1 I_2(\eta)-I_3(\eta)\right], \tag{B.1a}$$

and for $\eta_c\le\eta\le\eta_0$:

$$f_1(\eta)=\left(\frac{2}{kd+2}\right)^{\frac{k-1}{k}}\left(\frac{\eta_0^2}{\eta_0^2-\eta^2}\right)^{\frac{k-1}{k}} f_1(\eta_c)-\alpha_1\frac{2(kd+2)}{k\left(\eta_0^2-\eta^2\right)^{\frac{k-1}{k}}}\int_{\eta_c}^{\eta}\frac{I_0(\eta)}{\eta^{d-1}\left(\eta_0^2-\eta^2\right)^{\frac{1}{k}}}\,\mathrm{d}\eta, \tag{B.1b}$$

where

$$I_1(\eta)=\left(\frac{k\eta_0^2}{2(kd+2)(k+1)}\right)^{\frac{1}{k}}\int_{\eta}^{\eta_0}\lambda^{d-1}\left(\frac{\eta_0^2}{\eta_0^2-\lambda^2}\right)^{\frac{k-1}{k}}\mathrm{d}\lambda$$

$$I_2(\eta)=\int_0^{\eta}\frac{I_1(\lambda)}{\lambda^{d-1}\left(\eta_0^2-\lambda^2\right)^{\frac{1}{k}}}\,\mathrm{d}\lambda,\qquad I_3(\eta)=-\frac{\eta^2}{2(kd+2)}\left(\frac{k}{2(kd+2)(k+1)}\right)^{\frac{1}{k}}.$$

In order to eliminate the constant $f_1(0)$ we first multiply (B.1a–B.1b) by η^{d-1} and integrate the first from 0 to η_c and the second from η_c to η_0. After adding both, we can make use of the normalization (2.12) and obtain

$$0=f_1(0)\,I_7(\eta_c)+\left(\frac{2}{kd+2}\right)^{\frac{k-1}{k}} f_1(\eta_c)\,I_8(\eta_c)+\frac{2(kd+2)}{k}\left(\alpha_1[I_5(\eta_c)-I_6(\eta_c)]-I_4(\eta_c)\right), \tag{B.2}$$

where

$$I_4(\eta)=-\frac{1}{2(kd+2)}\left(\frac{k}{2(kd+2)(k+1)}\right)^{\frac{1}{k}}\int_0^{\eta}\frac{\lambda^{d+1}}{\left(\eta_0^2-\lambda^2\right)^{\frac{k-1}{k}}}\,\mathrm{d}\lambda, \tag{B.3}$$

$$I_5(\eta)=\int_0^{\eta}\frac{\lambda^{d-1}}{\left(\eta_0^2-\lambda^2\right)^{\frac{k-1}{k}}}\left[\int_0^{\lambda}\frac{I_1(\sigma)}{\sigma^{d-1}\left(\eta_0^2-\sigma^2\right)^{\frac{1}{k}}}\mathrm{d}\sigma\right]\mathrm{d}\lambda, \tag{B.4}$$

$$I_6(\eta)=\int_{\eta}^{\eta_0}\frac{\lambda^{d-1}}{\left(\eta_0^2-\lambda^2\right)^{\frac{k-1}{k}}}\left[\int_{\eta_c}^{\lambda}\frac{I_0(\sigma)}{\sigma^{d-1}\left(\eta_0^2-\sigma^2\right)^{\frac{1}{k}}}\mathrm{d}\sigma\right]\mathrm{d}\lambda, \tag{B.5}$$

and

$$I_7(\eta)=\left(\frac{k\eta_0^2}{2(kd+2)(k+1)}\right)^{-\frac{1}{k}} I_1(\eta),\qquad I_8(\eta)=\left(\frac{k\eta_0^2}{2(kd+2)(k+1)}\right)^{-\frac{1}{k}} I_0(\eta).$$

Thus, (B.2) together with (B.1a) at $\eta=\eta_c$ yields the formula

$$f_1(\eta_c)=\frac{8}{\eta_0^d B\left(\frac{d}{2},\frac{k+1}{k}\right)}\left(\alpha_1\left[I_7(\eta_c)I_2(\eta_c)-\eta_0^{\frac{2(k-1)}{k}}\left(I_5(\eta_c)-I_6(\eta_c)\right)\right]\right.$$
$$\left.+\eta_0^{\frac{2(k-1)}{k}} I_4(\eta_c)+\frac{kd\eta_0^2}{2(kd+2)^2}\left(\frac{k}{2(kd+2)(k+1)}\right)^{\frac{1}{k}} I_7(\eta_c)\right). \tag{B.6}$$

Appendix C. $O(\varepsilon^2)$ solution to the dipole problem

In order to determine $f_1(\eta_c)$, we observe first that (4.18a), (4.18b) can be written in the form

$$\frac{\mathrm{d}}{\mathrm{d}\eta}\left(y^{\frac{k-1}{k}}\eta^{\frac{-1}{k+1}}f_1(\eta)\right)=\frac{2(k+2)}{k}\left(\frac{\alpha_1}{y^{\frac{1}{k}}\eta^2}\int_0^\eta \lambda f_0-\frac{k}{2}\lambda^2\frac{\mathrm{d}f_0}{\mathrm{d}\lambda}\,\mathrm{d}\lambda-\frac{\eta\frac{\mathrm{d}f_0^{k+1}}{\mathrm{d}\eta}+f_0^{k+1}}{y^{\frac{1}{k}}\eta^2}\right), \tag{C.1a}$$

for $0\le\eta\le\eta_c$, and

$$\frac{\mathrm{d}}{\mathrm{d}\eta}\left(y^{\frac{k-1}{k}}\eta^{\frac{-1}{k+1}}f_1(\eta)\right)=-\frac{2(k+2)}{k}\frac{\alpha_1}{y^{\frac{1}{k}}\eta^2}\int_0^\eta \lambda f_0-\frac{k}{2}\lambda^2\frac{\mathrm{d}f_0}{\mathrm{d}\lambda}\,\mathrm{d}\lambda, \tag{C.1b}$$

for $\eta_c\le\eta\le\eta_0$, where we denote

$$y(\eta)=\eta_0^{\frac{k+2}{k+1}}-\eta^{\frac{k+2}{k+1}}.$$

We can now integrate (4.18a) from 0 to η and (4.18b) from η_c to η to obtain for $0\le\eta\le\eta_c$

$$\eta f_1(\eta)=\left(\frac{\eta_0}{y(\eta)}\right)^{\frac{k-1}{k}}\eta^{\frac{k+2}{k+1}}l_0+\frac{2(k+2)}{k}y(\eta)^{\frac{1-k}{k}}\eta^{\frac{k+2}{k+1}}\left(\alpha_1F_1(\eta)-F_2(\eta)\right), \tag{C.2a}$$

and for $0\le\eta\le\eta_c$

$$\eta f_1(\eta)=\left(\frac{y(\eta_c)}{y(\eta)}\right)^{\frac{k-1}{k}}\eta^{\frac{k+2}{k+1}}\eta_c^{\frac{-1}{k+1}}f_1(\eta_c)-\frac{2(k+2)}{k}y(\eta)^{\frac{1-k}{k}}\eta^{\frac{k+2}{k+1}}\alpha_1F_3(\eta), \tag{C.2b}$$

where we denote

$$l_0=\lim_{\eta\to0}\eta^{\frac{-1}{k+1}}f_1(\eta),$$

$$F_1(\eta)=\int_0^\eta\frac{1}{y(\lambda)^{\frac{1}{k}}\lambda^2}\left(\int_0^\lambda\sigma f_0-\frac{k}{2}\sigma^2\frac{\mathrm{d}f_0}{\mathrm{d}\sigma}\,\mathrm{d}\sigma\right)\mathrm{d}\lambda,$$

$$F_2(\eta)=\int_0^\eta\frac{\lambda\frac{\mathrm{d}f_0^{k+1}}{\mathrm{d}\lambda}+f_0^{k+1}}{y(\lambda)^{\frac{1}{k}}\lambda^2}\,\mathrm{d}\lambda,\quad F_3(\eta)=\int_{\eta_c}^\eta\frac{1}{y(\lambda)^{\frac{1}{k}}\lambda^2}\left(\int_\sigma^{\eta_0}\sigma f_0-\frac{k}{2}\sigma^2\frac{\mathrm{d}f_0}{\mathrm{d}\sigma}\,\mathrm{d}\sigma\right)\mathrm{d}\lambda.$$

We finally use the integral condition (4.8) after we integrate (C.2a) from 0 to η_c and (C.2b) from η_c to η_0 and adding the resulting parts, yielding

$$0=l_0\,G_1+\eta_c^{\frac{-1}{k+1}}f_1(\eta_c)\,H_1+\frac{2(k+2)}{k}(G_2-H_2), \tag{C.3}$$

where

$$G_1=\int_0^{\eta_c}\left(\frac{\eta_0}{y(\eta)}\right)^{\frac{k-1}{k}}\eta^{\frac{k+2}{k+1}}\,\mathrm{d}\eta,\quad G_2=\int_0^{\eta_c}y(\eta)^{\frac{1-k}{k}}\eta^{\frac{k+2}{k+1}}\left(\alpha_1F_1(\eta)-F_2(\eta)\right)\mathrm{d}\eta,$$

$$H_1=\int_{\eta_c}^{\eta_0}\left(\frac{y(\eta_c)}{y(\eta)}\right)^{\frac{k-1}{k}}\eta^{\frac{k+2}{k+1}}\,\mathrm{d}\eta,\quad H_2=\int_{\eta_c}^{\eta_0}y(\eta)^{\frac{1-k}{k}}\eta^{\frac{k+2}{k+1}}\alpha_1F_3(\eta)\,\mathrm{d}\eta.$$

Equation (C.3) together with (C.2a), evaluated at $\eta=\eta_c$, enable us to eliminate l_0 and to solve for $f_1(\eta_c)$:

$$f_1(\eta_c)=\frac{2(k+2)}{k}\eta_c^{\frac{1}{k+1}}\left(\frac{\eta_0^{\frac{1-k}{k}}\left(\alpha_1F_1(\eta_c)-F_2(\eta_c)\right)-\frac{G_2-H_2}{G_1}}{\left(y(\eta_c)/y(\eta_0)\right)^{\frac{k-1}{k}}+\frac{H_1}{G_1}}\right). \tag{C.4}$$

C.1. α_2 FOR THE FLOOD PROBLEM

The integrals, needed to find $f_1(\eta_c)$, turn out to be

$$F_1(\eta) = -\frac{\eta_0^{3/2}}{60}\log(1-z^3), \quad F_2(\eta) = -\frac{\eta^{3/2}}{72}, \quad \text{where } z = \left(\frac{\eta}{\eta_0}\right)^{1/2}.$$

$$F_3(\eta) = \frac{\eta_0^{3/2}}{60}\left[\sqrt{3}\arctan\left(\frac{2\zeta+1}{\sqrt{3}}\right) - \frac{1}{2}\log\left(\frac{(1-\zeta)^2}{1+\zeta+\zeta^2}\right) - \frac{3}{2\zeta^2} + \log(1-\zeta^3)\right]_v^z =: \frac{\eta_0^{3/2}}{60}\left(\phi(z) - \phi(v)\right), \quad \text{where } v = \left(\frac{\eta_c}{\eta_0}\right)^{1/2}.$$

$$G_1 = \frac{2}{5}\eta_c^{5/2}, \qquad H_1 = \frac{2}{5}\left(\eta_0^{5/2} - \eta_c^{5/2}\right).$$

$$H_2 = \frac{\alpha_1\eta_0^4}{150}\left(\sqrt{3}(1-v^5)\arctan\left(\frac{2v+1}{\sqrt{3}}\right) - \frac{3}{2}(1+v^5)\log(1+v+v^2) + \frac{3}{2}v^2\left(1+v+\frac{2}{5}v^3\right) + 3\left(\log(3) - \frac{6}{5}\right) - \phi(v)(1-v^5)\right).$$

$$G_2 = -\frac{\alpha_1\eta_0^4}{150}\left(\frac{v^5}{2}\log\left((1-v^3)^2\right) + \frac{1}{2}\log(1+v+v^2) - \frac{1}{2}\log\left((1-v^2)\right) - \sqrt{3}\arctan\left(\frac{2v+1}{\sqrt{3}}\right) - \frac{3}{2}v^2\left(1+\frac{2}{5}v^3\right) + \sqrt{3}\arctan\left(\frac{1}{\sqrt{3}}\right)\right) + \frac{\eta_c^4}{288}.$$

These yield the value $\alpha_2 = 0{\cdot}06773941887$.

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